Section 4.1

Linearization and Differentials

- Linearization
- Differentials
- Approximation and Relative Error

Given a function $y = f(x)$, the tangent line at $x = a$ is the line that just "touches" the curve at the point $(a, f(a))$, with slope $m = f'(a)$.

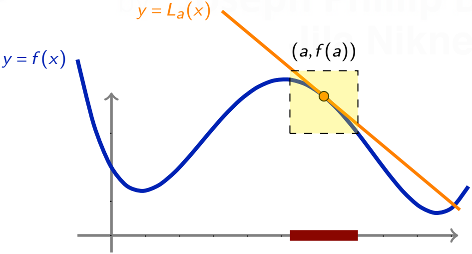
We can think of the tangent line equation itself as a function of x:

$$L_a(x) = f(a) + f'(a)(x - a)$$

Linear Approximation

For x-values near a, the tangent line L_a can be used to approximate the function $f(x)$. That is, if $|b - a|$ is small, then

$$f(b) \approx L_a(b) = f(a) + f'(a)(b - a)$$



Linear Approximation

Often, it is easier to calculate $L_a(b)$ than $f(b)$.

Example 1: Approximate $\sqrt{3.98}$.

Solution: We can use the function $f(x) = \sqrt{x}$ and the x-value $x = 4$ to approximate $\sqrt{3.98}$, because 4 is near 3.98 and we can easily calculate $\sqrt{4}$.

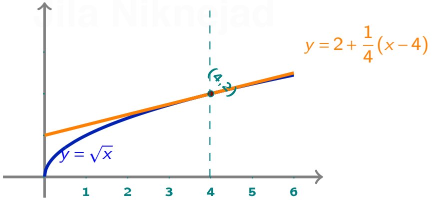
$$f(4) = \sqrt{4} = 2 \qquad f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

The tangent line to the graph of f at $(4,2)$ is $L_4(x) = 2 + \frac{1}{4}(x-4)$.

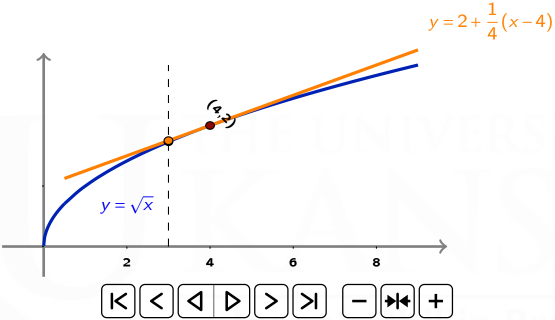
Therefore,

$$\sqrt{3.98} \approx L_4(3.98) = 1.995$$

This approximation is quite accurate: your calculator will tell you that $\sqrt{3.98} = 1.99499373\ldots$.



The linear approximation of $f(x) = \sqrt{x}$ at $x = 4$ gives approximations for x-values near $x = 4$. The farther x is from 4, the worse the approximation.



x	Δx	$L_4(x)$	$\sqrt{x}$	Percentage error
3	−1	1.7500	1.7321...	0.1%
3.9	−0.1	1.975	1.9748...	0.008%
4	0	2.000	2.0000...	0%
4.1	0.1	2.025	2.0248...	0.008%
5	1	2.250	2.2360...	0.62%
9	5	3.250	3.0000...	8.33%

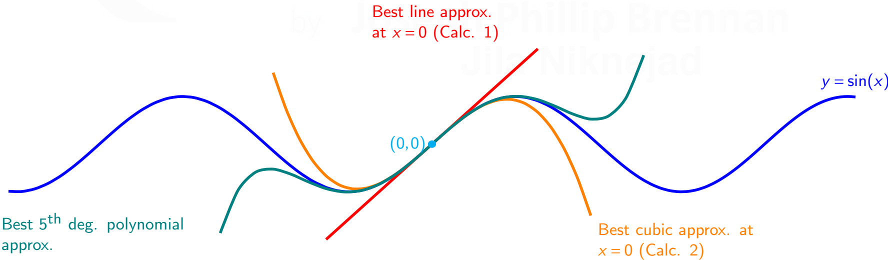
Linear Approximation Vs. Polynomial Approximation

Values of $f(x) = \sin(x)$ near $x = 0$ can be approximated through linearization.

$$f(0) = \sin(0) = 0 \qquad f'(0) = \cos(0) = 1$$

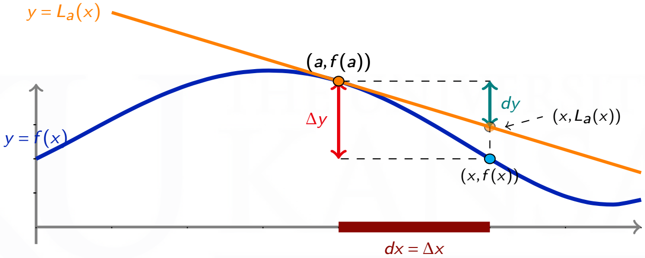
So $\sin(x) \approx 0 + 1(x - 0) = x$ for x-values near zero.
When approximating with tangent lines, the value used for approximation must be **very** close to $x = a$. For example, $\sin(6) \approx 6$ is a bad approximation.

Taylor Polynomials: Linearization can be generalized to higher degree polynomials, which typically have better approximations for values a large distance from $x = a$.



Increments and Differentials

We can finally say what dy and dx mean!

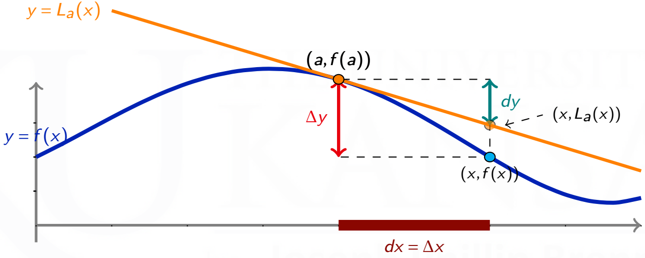


- In this figure, the value of x changes from a by Δx to $a + \Delta x$.
- The resulting change in $y = f(x)$ is $\Delta y = \Delta f = f(a + \Delta x) - f(a)$.
- The **differentials** dx and dy are the changes in the x- and y-coordinates of the tangent line:

$$dx = \Delta x \qquad\qquad dy = f'(a)\, dx$$

Error Estimation

Sometimes, we can only measure the value of x to within some error Δx. In that case, how accurate is the value of $f(x)$?



The maximum possible error in $f(x)$ is

$$\Delta f = \Delta y = f(a + \Delta x) - f(a)$$

which we can approximate by the differential df:

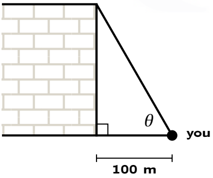
$$df = dy = f'(a)\Delta x = f'(a)\,dx.$$

Linear Approximation Error

If the value of the x is measured as $x = a$ with an error of $\pm\Delta x$, then Δf, the error in estimating $f(x)$, can be approximated as

$$\Delta f = f(x) - f(a) \quad \approx \quad f'(a)\Delta x = df.$$

Example 2: Standing 100 meters from a building, you estimate that your angle of inclination to the top of the building is $\theta = 60°$, with a possible error of 3°. How tall is the building? How accurate is your estimate?



Solution: The height of the building is $H(\theta) = 100\tan(\theta)$ for the **exact** angle θ. Plugging in $\theta = 60° = \frac{\pi}{3}$ radians gives $100\tan(\pi/3) \approx 173.2$ m.

The error in the angle measurement is $\Delta\theta = \pm 3° = \pm\frac{\pi}{60}$ radians, so the possible error in measuring H is

$$\Delta H \approx dH = H'\left(\frac{\pi}{3}\right)\Delta\theta = 100\sec^2\left(\frac{\pi}{3}\right)\left(\pm\frac{\pi}{60}\right) \approx \pm 20.94\,\text{m}.$$

Error Estimation: Example

Example 3: A gaming company produces cubical dice. For shipping purposes, each die must have volume $80\,\text{cm}^3$, with a tolerance of $\pm 2\,\text{cm}^3$. How long should each side be and how much variation can be allowed?

Solution: The volume of a cube is $V(x) = x^3$ where x is the side length. If the volume is to be $80\,\text{cm}^3$, then $x = \sqrt[3]{80} \approx 4.31\,\text{cm}$.

The variation in x must not cause V to vary more than $\pm 2\,\text{cm}^3$.
That is, we must choose dx so as to ensure that $|dV| \leq 2\,\text{cm}^3$.

$dV = V'(x)\,dx = 3x^2\,dx$

$2 \geq |dV| = 3(80^{2/3})\,|\Delta x| \quad \Rightarrow \quad |\Delta x| < \frac{2}{3}(80^{-2/3}) \approx 0.0359\ldots$

The side length should be 4.31 ± 0.0359 cm.

Percentage Error

Suppose that we measure some quantity as $x = a \pm \Delta x$ and then want to calculate $f(x)$. We often want to know how large the error Δf is *relative to the actual value of* $f(a)$.

That is, we want to look at the **percentage error**:

$$\text{Relative error } = \left|\frac{\text{error of } f}{\text{value of } f}\right| = \left|\frac{\Delta f}{f(a)}\right|$$

As before, we can use differentials to approximate percentage error:

$$\text{Relative error } = \left|\frac{\Delta f}{f(a)}\right| \approx \left|\frac{df}{f(a)}\right| = \left|\frac{f'(a)\,\Delta x}{f(a)}\right|.$$

(In order to convert this ratio to a percentage, multiply by 100%.)

Revisiting Example 2: Standing 100 meters from a building, you estimate that your angle of inclination to the top of the building is $\theta = 60°$, with a possible error of $3°$. What is the potential percentage error?

Relative error$= \frac{20.94}{173.2} \simeq 0.121$

Percentage error$= 12.1\%$

Percentage Error, Example (Optional)

Example 4: Using a ruler marked in millimeters, you measure the diameter of a circle as 6.4 cm. How large can the error in the calculated area of the circle be? What is the potential percentage error?

Solution: The area of a circle depends on diameter:

$$A(D) = \pi D^2/4 \qquad A'(D) = \pi D/2 \qquad A(6.4) = 32.17\,\text{cm}^2$$

The measured diameter might be in error by as much as $\Delta D = \pm 0.1$cm. That is, the diameter is $6.6 - 0.1 < D < 6.6 + 0.1$ cm.

$$dA = A'(6.4)\Delta D = \frac{\pi}{2}(6.4)(\pm 0.1) \approx \pm 1.01\,\text{cm}^2$$

Therefore the percentage error is

$$\frac{1.01\,\text{cm}^2}{32.17\,\text{cm}^2} = 0.0314 = 3.14\% \text{ of the measured area.}$$

Alternative Solution:

Now notice that, if I am using the measurement in radius, then the measured value is $r = 3.2$ and with the possible error: $\frac{0.1}{2} = 0.05$ mm. That is, the measurement is $3.2 - 0.05 < r < 3.2 + 0.05$.

Relate the area of circle with radius:

$$A(r) = \pi r^2 \qquad A'(r) = 2\pi r \qquad A(3.2) \approx 32.17\,\text{cm}^2$$

Now

$$\Delta A \approx |dA| = |A'(3.2)|.|\Delta x| < (6.4\pi)(|\pm 0.05|) \approx 1.01\,\text{cm}^2$$

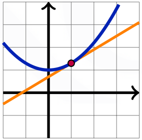
Therefore the percentage error is

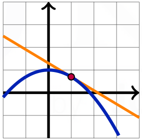
$$\frac{1.01\,\text{cm}^2}{32.17\,\text{cm}^2} = 0.0314 = 3.14\% \text{ of the measured area.}$$

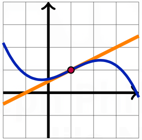
The Effect of Concavity

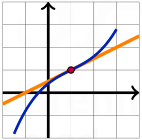
The accuracy of an approximation using a tangent line is affected by the concavity of the curve. At a point $(a, f(a))$,

concave up	$\Leftrightarrow$	$f''(a) > 0$	$\Leftrightarrow$	graph lies above tangent line	$\Leftrightarrow$	underestimate
concave down	$\Leftrightarrow$	$f''(a) < 0$	$\Leftrightarrow$	graph lies below tangent line	$\Leftrightarrow$	overestimate


Concave Up


Concave Down


Inflection Point


Inflection Point

The greater the absolute value of $f''(a)$, the more the graph diverges from the tangent line.

The Effect of Concavity, Example (Optional)

Example 5: Use a suitable linear approximation to estimate ln(1.1). Is your estimate greater than, less than, or equal to the actual value?

Solution: Linearize $f(x) = \ln(x)$ at $x = 1$ to approximate ln(1.1):

$$f(1) = 0 \qquad f'(x) = \frac{1}{x} \qquad f'(1) = 1 \qquad L_1(x) = x - 1$$

So $\ln(1.1) \approx L_1(1.1) = \boxed{0.1}$.

Since $f''(x) = -x^{-2}$, the graph of $\ln(x)$ function is concave down everywhere.

Therefore, the function lies **below** any tangent line and the estimation is an **over**-estimate.

In fact $\ln(1.1) = 0.0953101\ldots < 0.1$.